

COMPLEX MONGE-AMPERE OPERATORS VIA PSEUDO-ISOMORPHISMS: THE WELL-DEFINED CASES

TUYEN TRUNG TRUONG

ABSTRACT. Let X and Y be compact Kähler manifolds of dimension 3. A bimeromorphic map $f : X \rightarrow Y$ is pseudo-isomorphic if $f : X - I(f) \rightarrow Y - I(f^{-1})$ is an isomorphism.

Let $T = T^+ - T^-$ be a current on Y , where T^\pm are positive closed $(1, 1)$ currents which are smooth outside a finite number of points. We assume that the following condition is satisfied:

Condition 1. For every curve C in $I(f^{-1})$, then in cohomology $\{T\} \cdot \{C\} = 0$.

Then, we define a natural push-forward $f_*(\varphi dd^c u \wedge f^*(T))$ for a quasi-psh function u and a smooth function φ on Y . We show that this pushforward satisfies a Bedford-Taylor's monotone convergence type.

Assume moreover that the following two conditions are satisfied

Condition 2. The signed measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$.

Condition 3. For every curve C in $I(f^{-1})$, the measure $T \wedge [C]$ has no Dirac mass.

Then, we define a Monge-Ampere operator $MA(f^*(T)) = f^*(T) \wedge f^*(T) \wedge f^*(T)$ for $f^*(T)$. We show that this Monge-Ampere operator satisfies several continuous properties, including a Bedford-Taylor's monotone convergence type when T is positive. The measures $MA(f^*(T))$ are in general quite singular. Also, note that it may be not possible to define $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$.

1. INTRODUCTION

Let X and Y be compact Kähler manifolds of dimension 3. A bimeromorphic map $f : X \rightarrow Y$ is pseudo-isomorphic if the map $g = f|_{X - I(f)} : X - I(f) \rightarrow Y - I(f^{-1})$ is an isomorphism. Here $I(f)$ and $I(f^{-1})$ are the indeterminate sets of f and f^{-1} , both have dimensions at most 1. (In fact, Bedford-Kim [3] showed that if $I(f)$, and hence $I(f^{-1})$, is non-empty then it must be of pure dimension 1). We let $\Gamma_g \subset (X - I(f)) \times (Y - I(f^{-1}))$ be the graph of g , and $\Gamma_f =$ the closure of Γ_g in $X \times Y$ the graph of f . Let $\pi_1, \pi_2 : X \times Y \rightarrow X, Y$ be the natural projections, and occasionally we use the same notations for the restrictions to Γ_g, Γ_f .

Given a meromorphic map $f : X \rightarrow Y$ and a smooth closed $(1, 1)$ form θ on Y , the pullback $f^*(\theta)$ is well-defined as a $(1, 1)$ current, however, in general is singular on $I(f)$. To see an explicit example, consider the simple map $J : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ given by the formula $J[x_0 : x_1 : x_2 : x_3] = [1/x_0 : 1/x_1 : 1/x_2 : 1/x_3]$. If u is a smooth function then $J^*dd^c u$ will have singularities of the form $1/(x_k^2 \bar{x}_l^2)$ near the curves of indeterminacy $x_i = x_j = 0$. Therefore, a priori it is not clear whether we can define the Monge-Ampere operator $MA(J^*dd^c u) = J^*dd^c u \wedge J^*dd^c u \wedge J^*dd^c u$ in a reasonable manner.

In [13], we show that if $f : X \rightarrow Y$ is a pseudo-isomorphism in dimension 3, and θ is a smooth closed $(1, 1)$ form on Y such that in cohomology

$$\{\theta\} \cdot \{C\} = 0,$$

for every curve C in $I(f^{-1})$, then we have a well-defined Monge-Ampere operator $MA(f^*(\theta))$. For example, the map J above is not yet pseudo-isomorphic, but if we let $X \rightarrow \mathbb{P}^3$ be the blowup at the 4 points $e_0 = [1 : 0 : 0 : 0]$, $e_1 = [0 : 1 : 0 : 0]$, $e_2 = [0 : 0 : 1 : 0]$, $e_3 = [0 : 0 : 0 : 1]$, then the induced map J_X is a pseudo-isomorphism. Hence, for any smooth function u on $Y = X$, we can define the Monge-Ampere operator $J_X^* dd^c u$ in a reasonable and consistent way, even though as we saw above this $(1, 1)$ current is quite singular. Note that if we write $dd^c u = \alpha^+ - \alpha^-$, where α^\pm are positive closed smooth $(1, 1)$ forms, there may be no reasonable and consistent way to define the wedge products $J_X^*(\alpha^\pm) \wedge J_X^*(\alpha^\pm) \wedge J_X^*(\alpha^\pm)$. For such an intersection to be well-defined, we may want to show that $J_X^*(\alpha^\pm)$ have locally bounded potentials near $I(J_X)$. However, we have the following result

Lemma 1.1. *Let C be an irreducible curve in $I(J_X)$. Then $J_X(C) = D$ is another irreducible curve in $I(J_X)$. If ω is a positive closed smooth $(1, 1)$ form on X such that $\{\omega\} \cdot \{D\} > 0$, then the local potentials of $J_X^*(\omega)$ are unbounded near C .*

Proof. That $J_X(C) = D$ is an irreducible curve in $J_X(C)$ can be checked directly (see the last Section in [13]). Now we prove the claim about the unboundedness of the local potentials of $J_X^*(\omega)$ near C . Assume otherwise. Then by Bedford-Taylor's results [4], the wedge intersection of currents $J_X^*(\omega) \wedge [C]$ is well-defined as a positive measure on X . In particular, in cohomology

$$0 \leq \{J_X^*(\omega) \wedge [C]\} = \{J_X^*(\omega)\} \cdot \{C\} = \{\omega\} \cdot (J_X)_* \{C\}.$$

However, we can check that in cohomology $(J_X)_* \{C\} = -\{D\}$ (see the last section in [13]). Hence we obtain

$$\{\omega\} \cdot (J_X)_* \{C\} = -\{\omega\} \cdot \{D\} < 0.$$

by assumption. This is a contradiction. \square

The purpose of this short note is to extend the Monge-Ampere operator $MA(f^*(T)) = f^*(T) \wedge f^*(T) \wedge f^*(T)$ in [13] to currents T which can be singular on a finite number of points. The points are allowed to be in $I(f^{-1})$. The main motivation for this is that given a pscf cohomology class $\eta \in H^{1,1}(X)$, it may not be able to find a positive closed smooth form θ in that class, while if we allow a mild singularity there may be a positive closed $(1, 1)$ current in the class of η with that singularity. Moreover, if we allow more singularity for T , then the current $f^*(T)$ may be more singular and hence it makes it more difficult to define $MA(f^*(T))$.

We show that the Monge-Ampere operator so defined satisfies various continuous properties, see in particular Theorems 2.8 and 2.13, Lemmas 2.7 and 2.9, and the last subsection of the paper. In the proof of the continuous properties, we will use the following approximation of positive closed smooth $(1, 1)$ currents, due to Demailly [6].

Definition 1.2. Let Y be a compact Kähler manifold with a Kähler form ω_Y . Let $T = \alpha + dd^c u$ be a positive closed $(1, 1)$ current on Y , where α is a smooth closed $(1, 1)$ form

and u is a quasi-psh function. Let u_j be a sequence of smooth quasi-psh functions on Y decreasing to u such that $\alpha + dd^c u_j + \epsilon \omega_Y \geq 0$ for all j , here $\epsilon > 0$ is a positive constant. Then we say that $\alpha + dd^c u_j$ is a good approximation of $T = \alpha + dd^c u$.

Here we summarize the main results.

Theorem 1.3. *Let $f : X \rightarrow Y$ be a pseudo-isomorphism in dimension 3. Let $T = T^+ - T^-$ be a difference of two positive closed currents $(1, 1)$ currents on Y , both are smooth outside a finite number of points. These points are allowed to be in $I(f^{-1})$.*

Assume that for every curve C in $I(f^{-1})$ we have in cohomology $\{T\} \cdot \{C\} = 0$.

We write $f^(T) = \Omega + dd^c u$, where Ω is a smooth closed $(1, 1)$ form and $u = u^+ - u^-$ is a difference of two quasi-psh functions.*

1) (Bedford-Taylor's monotone convergence.) *Let u_j^\pm be smooth quasi-psh functions decreasing to u^\pm . Then for any smooth function φ on X , the following limit exists*

$$\lim_{j \rightarrow \infty} f_*(\varphi(\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge f^*(T)).$$

We denote the limit by $f_(\varphi f^*(T) \wedge f^*(T))$.*

2) *Let S be a smooth closed $(1, 1)$ form on Y . Let u_j^\pm be as in 1). Then*

$$\lim_{j \rightarrow \infty} \int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge f^*(T) = \int_Y S \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

3) *Assume further that T satisfies the following two conditions:*

i) *The measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$.*

ii) *For each curve C in $I(f^{-1})$ then the measure $T \wedge [C]$ has no Dirac mass.*

Then there is a natural and well-defined wedge intersection of currents $T \wedge f_(\varphi f^*(T) \wedge f^*(T))$. In view of 2) above, we define the Monge-Ampere operator $MA(f^*(T))$ by the formula*

$$\langle MA(f^*(T)), \varphi \rangle := \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

4) *Assumptions are as in 3). Assume further that T is a positive current. We write $T = \alpha + dd^c v$, where α is a smooth closed $(1, 1)$ form and v is a quasi-psh function. Let $\alpha + dd^c v_n$ be a good approximation of $T = \alpha + dd^c v$, in the sense of Definition 1.2. Then for any smooth function φ we have*

$$\lim_{n \rightarrow \infty} \int_Y (\alpha + dd^c v_n) \wedge f_*(\varphi f^*(T) \wedge f^*(T)) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

In other words, we have a double Bedford-Taylor's monotone type convergence

$$\lim_{n \rightarrow \infty} \lim_{j \rightarrow \infty} \int_X \varphi f^*(\alpha + dd^c v_n) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

5) *Assumptions are as in 4). Assume moreover that $f^*(T \wedge T)$ has no mass on $I(f)$ (for example if T is smooth near $I(f^{-1})$). If $\Omega + dd^c u_j$ is a good approximation of $f^*(T) = \Omega + dd^c u$ in the sense of Definition 1.2, then*

$$\lim_{j \rightarrow \infty} \int_X \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

6) *Assumptions are as in 3). If T is smooth or positive then $MA(f^*(T)) = f^*(T \wedge T \wedge T)$. Here, since the measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$, the pullback $f^*(T \wedge T \wedge T)$ is well-defined.*

Remark 1.4. In 6) of Theorem 1.3, a priori the measure $f^*(T \wedge T \wedge T)$ is quite singular near $I(f)$, even if T is smooth. Also, note that there may be no reasonable and consistent manner to define the terms $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$, so we need to define $f^*(T) \wedge f^*(T) \wedge f^*(T)$ directly. See Lemma 1.1 and the discussion before it.

2. DEFINITION OF THE MONGE-AMPERE OPERATOR

We will consider the following class of currents

Definition 2.1. Class (\mathcal{A}) . A closed $(1, 1)$ current T is in class (\mathcal{A}) if $T = T^+ - T^-$ where T^\pm are positive closed $(1, 1)$ currents which are smooth outside a finite number of points.

Remark 2.2. The essential property that we need in the above definition is that in $W - A$, here W is an open neighborhood of $I(f^{-1})$ and A is a finite set, the currents T^\pm are smooth (in fact, continuous is enough). Outside $W - A$, T^\pm may have mild singularity such that $T \wedge T \wedge T$ is well-defined. For example, following Bedford-Taylor [4], we need only to require that T^\pm have locally bounded potentials.

Remark 2.3. That T^\pm may have singular points on $I(f^{-1})$ makes it difficult to define the individual wedge products of currents $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$. This is because the preimage of a point on $I(f^{-1})$ may be a whole curve on $I(f)$. So a priori $f^*(T^\pm)$ may be singular on a whole curve contained in $I(f)$, see for example Lemma 1.1. Hence, in the below, we will define $f^*(T) \wedge f^*(T) \wedge f^*(T)$ directly, not via the wedge products $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$.

We will consider the following three conditions

Condition 1. For every curve C in $I(f^{-1})$, then in cohomology $\{T\} \cdot \{C\} = 0$.

Condition 2. The signed measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$.

Condition 3. For every curve C in $I(f^{-1})$, the measure $T \wedge [C]$ has no Dirac mass.

Remark 2.4. If T is in Class (\mathcal{A}) , then the measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$, except possibly a finite number of points on $I(f^{-1})$ where T is not smooth. Hence Condition 2 is equivalent to that $T \wedge T \wedge T$ has no Dirac masses at these points.

Remark 2.5. If T is smooth then T satisfies both Conditions 2 and 3.

If T is a positive closed $(1, 1)$ current in Class (\mathcal{A}) and satisfies Condition 1, then it automatically satisfies Condition 3. Because in this case the wedge product of currents $T \wedge [C]$ is well-defined as a positive measure, and the total mass is $\{T\} \cdot \{C\}$. However, if T is not positive then this implication is not automatic.

Assume that the Monge-Ampere operator $MA(f^*(T)) = f^*(T) \wedge f^*(T) \wedge f^*(T)$ is well-defined. Then, formally, for a smooth function φ on X we have

$$(2.1) \quad \int_X \varphi f^*(T) \wedge f^*(T) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)),$$

provided that both wedge intersections of currents $f_*(\varphi f^*(T) \wedge f^*(T))$ and $T \wedge f_*(\varphi f^*(T) \wedge f^*(T))$ are well-defined. The remaining of this note is to define these under the assumption that T is in Class \mathcal{A} and satisfies Conditions 1, 2 and 3.

Remark 2.6 (Justification for the approach.). Under Condition 1, we showed in [13] that $f^*(T) \wedge f^*(T) = f^*(T \wedge T)$, so one may attempt to define $MA(f^*(T))$ in a different way

$$(2.2) \quad \int_X \varphi f^*(T) \wedge f^*(T) \wedge f^*(T) = \int_X f_*(\varphi f^*(T)) \wedge T \wedge T.$$

At a first look, this approach seems to have equal footing with our approach in Equation (2.1). To justify what approach is more reasonable, let us consider a more general problem. Assume that S is another $(1,1)$ current which is smooth, and we want to define $f^*(S) \wedge f^*(T) \wedge f^*(T)$.

Our approach in Equation (2.1) is to define

$$\int_X \varphi f^*(S) \wedge f^*(T) \wedge f^*(T) := \int_Y S \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

The right hand side of the above expression is well-defined, since S is smooth, provided that the current $f_*(\varphi f^*(T) \wedge f^*(T))$ is defined. Moreover, the equality is justified by proving a continuity property, see Lemma 2.7 below.

The approach in Equation 2.2 is to define either

$$\int_X \varphi f^*(S) \wedge f^*(T) \wedge f^*(T) := \int_Y f_*(\varphi f^*(S)) \wedge T \wedge T,$$

or

$$\int_X \varphi f^*(S) \wedge f^*(T) \wedge f^*(T) := \int_Y f_*(\varphi f^*(T)) \wedge S \wedge T.$$

Since T may not be smooth, the equalities between the two sides of the above two expressions are not justified, if φ is not a constant.

From this simple consideration, we see that the definition in Equation (2.1) is more reasonable. Moreover, we will show later that if either T is smooth or positive, then the definitions in Equations (2.1) and (2.2) are the same.

Now we state and prove the continuous property referred to in the above remark.

Lemma 2.7. (*Bedford-Taylor's monotone convergence.*) Assume S is a smooth closed $(1,1)$ form and T is a current in the class (\mathcal{A}) and satisfies Condition 1. We write $f^*(T) = \Omega + dd^c u$, where Ω is a smooth closed $(1,1)$ form and $u = u^+ - u^-$ is the difference of two quasi-psh functions. Let u_j^\pm be a sequence of smooth quasi-psh functions decreasing to u^\pm . We denote $u_j = u_j^+ - u_j^-$. Then

$$\lim_{j \rightarrow \infty} \int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y S \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)).$$

Here the current $f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$ is defined in Equation (2.3) below.

Proof. First, we show that for each j

$$\int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y S \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)).$$

Here both sides are well-defined, since φ , S , Ω and u_j are smooth. The term $f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T))$ is defined as follows, by Meo's results:

$$f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = (\pi_2)_*(\pi_1^*(\varphi(\Omega + dd^c u_j)) \wedge \pi_1^*(f^*(T)) \wedge [\Gamma_f]).$$

Now we can approximate $f^*(T)$ by smooth closed $(1, 1)$ forms $\gamma_n = \gamma_n^+ - \gamma_n^-$. Here γ_n^\pm positive closed smooth $(1, 1)$ forms with uniformly bounded masses, and converges locally uniformly on $X - I(f)$ to $f^*(T)$.

Then it can be seen, by dimension reason (see for example the proof of Lemma 5 in [12]), that

$$\int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \lim_{n \rightarrow \infty} \int_Y \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-).$$

Since all φ , S , Ω , u_j and γ_n^\pm are all smooth, we have

$$\int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-) = \int_Y S \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-)).$$

Now

$$f_*(\varphi(\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-)) = (\pi_2)_*(\pi_1^*(\varphi)\pi_1^*(\Omega + dd^c u_j) \wedge \pi_1^*(\gamma_n^+ - \gamma_n^-) \wedge [\Gamma_f]).$$

The limit when $n \rightarrow \infty$ of the right hand side is $(\pi_2)_*(\pi_1^*(\varphi)\pi_1^*(\Omega + dd^c u_j) \wedge \pi_1^*(f^*(T)) \wedge [\Gamma_f])$. This is because the limit of $\pi_1^*(\varphi)\pi_1^*(\Omega + dd^c u_j) \wedge \pi_1^*(\gamma_n^+ - \gamma_n^-) \wedge [\Gamma_f]$ is $\pi_1^*(\varphi)\pi_1^*(\Omega + dd^c u_j) \wedge \pi_1^*(f^*(T)) \wedge [\Gamma_f]$.

Therefore the claim is proved. Using this claim and part 2) of Theorem 2.8 below, the lemma follows. \square

2.1. Definition of the current $f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$. We denote by $(\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$ the extension by zero of the current $\pi_1^*(f^*(T)) \wedge [\Gamma_g]$ (the latter has bounded mass by Meo's result [10]). Let u be a quasi-psh function on X . Theorem 1.2 in [13] shows that the current $\pi_1^*(uf^*(T)) \wedge [\Gamma_g]$ has bounded mass, and we let $(\pi_1^*(uf^*(T)) \wedge [\Gamma_g])^o$ denote its extension by zero. In [13], we defined

$$(2.3) \quad f_*(\varphi dd^c u \wedge f^*(T)) := (\pi_2)_*(\pi_1^*(\varphi) \wedge dd^c(\pi_1^*(uf^*(T)) \wedge [\Gamma_g])^o)$$

We now prove a Bedford-Taylor's monotone convergence theorem for this operator.

Theorem 2.8. *Assume that T is in Class (\mathcal{A}) and satisfies the Condition 1. Then*

1) *If u is a smooth quasi-psh function on X , we have*

$$f_*(\varphi dd^c u \wedge f^*(T)) = (\pi_2)_*(\pi_1^*(\varphi) \wedge dd^c \pi_1^*(uf^*(T)) \wedge [\Gamma_f]).$$

The right hand side above is the (correct) usual definition in the case u is smooth.

2) *Let u be a quasi-psh function on X , and let u_j be a sequence of smooth quasi-psh functions decreasing to u . Then*

$$\lim_{j \rightarrow \infty} f_*(\varphi dd^c u_j \wedge f^*(T)) = f_*(\varphi dd^c u \wedge f^*(T)).$$

Proof. 1) A modification of the proof of Theorem 1.3 in [13] shows that

$$\pi_1^*(f^*(T)) \wedge [\Gamma_f] = (\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o + \sum_j \lambda_j [V_j].$$

Here $\lambda_j \geq 0$ is a constant, and V_j are varieties of dimension 2 contained in $\Gamma_f - \Gamma_g$. Moreover, $\pi_2(V_j)$ are contained in the finite set of singular points of T .

Since u is smooth, it is not difficult to check that

$$(2.4) \quad \pi_1^*(u)(\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o = (\pi_1^*(u)\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o.$$

Since we will use similar arguments later on, we give here a detailed proof. Using $T = T^+ - T^-$, we may assume that T is positive. We may also assume that $0 \geq u \geq -M$. Then $\pi_1^*(u)(\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$ is bounded between the two negative currents 0 and $-M(\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$. Both these currents have no mass on $\Gamma_f - \Gamma_g$, so is $\pi_1^*(u)(\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$. On Γ_g , $\pi_1^*(u)(\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$ equals $(\pi_1^*(u)\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$, and the current $(\pi_1^*(u)\pi_1^*(f^*(T)) \wedge [\Gamma_g])^o$ has no mass on $\Gamma_f - \Gamma_g$ by definition. Therefore, the two currents in Equation (2.4) are the same on Y .

For any j , since $\pi_2(V_j)$ is a point, by the dimension reason we see immediately that

$$(\pi_2)_*(\pi_1^*(\varphi)dd^c\pi_1^*(u) \wedge [V_j]) = 0.$$

Therefore we obtain

$$(\pi_2)_*(\pi_1^*(\varphi) \wedge dd^c\pi_1^*(uf^*(T)) \wedge [\Gamma_f]) = (\pi_2)_*(\pi_1^*(\varphi) \wedge dd^c(\pi_1^*(uf^*(T)) \wedge [\Gamma_g])^o),$$

and the latter was defined to be $f_*(\varphi dd^c u \wedge f^*(T))$ in Equation (2.3).

2) From Equation (2.4), it suffices to show that

$$\lim_{j \rightarrow \infty} (\pi_1^*(u_j f^*(T)) \wedge [\Gamma_g])^o = (\pi_1^*(u f^*(T)) \wedge [\Gamma_g])^o.$$

The proof of this is similar to that used to prove Equation (2.4). We can assume that T is positive, all u_j and u are negative. Let R be one cluster point of the left hand side. Then R is negative, $R \geq$ the right hand side, and on Γ_g then $R =$ the right hand side. Since the right hand side has no mass on $\Gamma_f - \Gamma_g$ by definition, we conclude that $R =$ the right hand side. \square

We write $f^*(T) = \Omega + dd^c u$, where Ω is a smooth closed $(1,1)$ form, and $u = u^+ - u^-$ is a difference of two quasi-psh functions. By Theorem 2.8, the pushforward

$$(2.5) \quad f_*(\varphi f^*(T) \wedge f^*(T)) := f_*(\varphi \Omega \wedge f^*(T)) + f_*(\varphi dd^c u^+ \wedge f^*(T)) - f_*(\varphi dd^c u^- \wedge f^*(T))$$

is well-defined. Moreover, if u_j^\pm is a sequence of smooth quasi-psh functions decreasing to u^\pm then

$$(2.6) \quad \lim_{j \rightarrow \infty} f_*(\varphi(\Omega + dd^c(u_j^+ - u_j^-)) \wedge f^*(T)) = f_*(\varphi f^*(T) \wedge f^*(T)).$$

This Bedford-Taylor's monotone convergence type implies the following

Lemma 2.9. *The definition in Equation (2.5) is independent of the choice of Ω and u in $f^*(T) = \Omega + dd^c u$.*

2.2. Definition of the current $T \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$. Let $f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$ be the current defined in the previous subsection. We now define the intersection $T \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$. Without loss of generality we may assume that $0 \leq \varphi \leq 1$.

We recall that from Theorem 2.8, if $u = u^+ - u^-$ where u^\pm are quasi-psh functions, and u_j^\pm are smooth quasi-psh functions decreasing to u^\pm then

$$\begin{aligned} f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) &= \lim_{j \rightarrow \infty} f_*(\varphi(\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge f^*(T)) \\ &= \lim_{j \rightarrow \infty} (\pi_2)_*(\pi_1^*(\varphi)\pi_1^*(\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge \pi_1^*f^*(T) \wedge [\Gamma_f]). \end{aligned}$$

While the sequence $\pi_1^*(\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge \pi_1^*f^*(T) \wedge [\Gamma_f]$ may not have a limit, it is a compact sequence and we let $R^+ - R^-$ be a cluster point. Here R^\pm are positive closed currents of bidimension (1,1) supported in Γ_f . By the result discussed in the previous paragraph, we have

$$f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = (\pi_2)_*(\pi_1^*(\varphi)R^+ - \pi_1^*(\varphi)R^-).$$

Since we assumed that $0 \leq \varphi \leq 1$, we have

$$0 \leq (\pi_2)_*(\pi_1^*(\varphi) \wedge R^\pm) \leq (\pi_2)_*(R^\pm)$$

Remark 2.10. Note that, under Condition 1 and the assumption that T is in class (\mathcal{A}) , then

$$(\pi_2)_*(R^+ - R^-) = f_*(f^*(T) \wedge f^*(T)) = f_*(f^*(T \wedge T)) = T \wedge T$$

has no mass on $I(f^{-1})$. Here we used that $f_*f^* = Id$ on positive closed (1,1) and (2,2) currents, see Theorem 1 in [11]. However, each $(\pi_2)_*(R^\pm)$ may have mass on $I(f^{-1})$. Therefore, if φ is not a constant, $(\pi_2)_*(\pi_1^*(\varphi)R^+ - \pi_1^*(\varphi)R^-)$ may have mass on $I(f^{-1})$.

Since the currents $(\pi_2)_*(\pi_1^*(\varphi) \wedge R^\pm)$ are positive DSH currents in the sense in Dinh-Sibony [7, 8], they are \mathbb{C} -flat in the sense of Bassanelli [2]. By Federer-type \mathbb{C} -flatness theorem (Theorem 1.24 in [2]), the restrictions of $(\pi_2)_*(\pi_1^*(\varphi) \wedge R^\pm)$ to $I(f^{-1})$ are well-defined as a current on $I(f^{-1})$.

Note that on $Y - I(f^{-1})$, then $f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = f_*(\varphi)T \wedge T$. Let $(f_*(\varphi)T \wedge T)^o$ be the extension by zero of this current from $Y - I(f^{-1})$ to Y . From the discussion above, and taking the bidimension of the various currents into consideration, we obtain the following result

Lemma 2.11.

$$f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = (f_*(\varphi)T \wedge T)^o + \sum_j (\chi_j^+ - \chi_j^-)[C_j].$$

Here C_j are irreducible components of dimension 1 of $I(f^{-1})$, and χ_j^\pm are bounded positive measurable functions on C_j .

Let A be the finite set where T is not smooth. Since $f_*(\varphi)$ is a difference of two quasi-psh functions and T^\pm are continuous on $Y - A$, by results in Fornaess-Sibony [9] and Demailly [5] (Section 4, Chapter 3), the current $f_*(\varphi)T \wedge T$ is well-defined on $Y - A$. Moreover, a monotone convergence property holds. Therefore, since $T^\pm \wedge T^\pm$ are positive closed currents with no mass on $I(f^{-1})$, an argument similar to that in the proof of Equation (2.4) concludes that $f_*(\varphi)T \wedge T$ has no mass on $I(f^{-1}) - A$. By dimension reason, we

see that $f_*(\varphi)T \wedge T$ extends as a current on Y . The extension current is the same as the current $(f_*(\varphi)T \wedge T)^o$ defined before Lemma 2.11.

By Lemma 2.11, to define $T \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$, it is enough to define $T \wedge (f_*(\varphi)T \wedge T)^o$ and $T \wedge \chi_j^\pm[C_j]$ for each j . We note that $T \wedge T \wedge T = \mu^+ - \mu^-$, where μ^\pm are positive measures which are smooth on $Y - A$. If Condition 2 is satisfied, then we can choose μ^\pm to have no mass on A . Similarly, $T \wedge [C_j]$ is a difference of two positive measures, which we can take to have no Dirac mass if Condition 3 is satisfied.

The following continuous property is a simple result in measure theory. For completeness, we include a proof of it here.

Lemma 2.12. *Assume that T is in Class (A) and Condition 2) is satisfied.*

Let γ_n be a sequence of uniformly continuous functions on Y which converges to $f_(\varphi)$ as currents. Moreover, assume that $\gamma_n = f_*(\varphi)$ on an open set W with $W \cap I(f^{-1}) = \emptyset$, such that T is smooth on $X - W - I(f^{-1})$. Then the sequence $T \wedge (\gamma_n T \wedge T)^o = \gamma_n T \wedge T \wedge T$ converges to $f_*(\varphi)(\mu^+ - \mu^-)$. Here the measure $f_*(\varphi)(\mu^+ - \mu^-)$ is well-defined on $Y - A$, and is defined to be 0 on the finite set A .*

A similar result holds when we consider the measures $T \wedge [C_j]$ and the functions χ_j^\pm .

Proof. Since γ_n is smooth, and $T \wedge T$ has no mass on $I(f^{-1})$, we have $(\gamma_n T \wedge T)^o = \gamma_n T \wedge T$.

Since μ^\pm are positive smooth measures on $Y - A$, we have

$$\lim_{n \rightarrow \infty} \gamma_n T \wedge T \wedge T = f_*(\varphi)(\mu^+ - \mu^-),$$

on $Y - A$.

Since μ^\pm are positive measures with no mass on A , any cluster point of $\gamma_n \mu^\pm$, which is bounded by μ^\pm , also has no mass on A . Therefore we obtain

$$\lim_{n \rightarrow \infty} \gamma_n T \wedge T \wedge T = f_*(\varphi)(\mu^+ - \mu^-),$$

on all of A . □

By Lemma 2.12, the wedge intersection $T \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T))$ is well-defined using a continuous property.

2.3. The case T is smooth or positive. We now show that in case T is smooth or positive then the Monge-Ampere in our approach Equation (2.1) and the approach in Equation (2.2) are the same.

We first consider the case where T is smooth. Then, by Theorem 2.8, the Monge-Ampere operator $MA(f^*(T)) = f^*(T) \wedge f^*(T) \wedge f^*(T)$ defined in Equation (2.1) is

$$\langle MA(f^*(T)), \varphi \rangle = \lim_{j \rightarrow \infty} \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T),$$

where u_j is an appropriate sequence of smooth functions converging to u . Since T satisfies Condition 1, we have $f^*(T) \wedge f^*(T) = f^*(T \wedge T)$. Then

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_X \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T) &= \lim_{j \rightarrow \infty} \int_X \varphi (\Omega + dd^c u_j) \wedge f^*(T \wedge T) \\ &= \lim_{j \rightarrow \infty} \int_X f_*(\varphi(\Omega + dd^c u_j)) \wedge T \wedge T. \end{aligned}$$

Since T is smooth on Y and $\Omega + dd^c u_j \rightarrow f^*(T)$, the limit of the sequence of measures $f_*(\varphi(\Omega + dd^c u_j)) \wedge T \wedge T$ is exactly $f_*(\varphi f^*(T)) \wedge T \wedge T$. It is also the same as $f_*(\varphi) f_*(f^*(T)) \wedge T \wedge T = f_*(\varphi) T \wedge T \wedge T$. Here we use that $f_* f^* = Id$ on positive closed $(1, 1)$ and $(2, 2)$ currents, by Theorem 1 in [11]. Thus the proof for the case T is smooth is completed.

Now we consider the case T is positive. We have the following

Theorem 2.13. *Assume T is a positive closed $(1, 1)$ current in Class (\mathcal{A}) and satisfies Condition 1. Then*

1)

$$(2.7) \quad f_*(\varphi f^*(T) \wedge f^*(T)) = f_*(\varphi) T \wedge T.$$

2) Assume moreover that T satisfies Conditions 2), 3). We write $T = \alpha + dd^c v$, where α is a smooth closed $(1, 1)$ form and v is a quasi-psh function. Let $\alpha + dd^c v_n$ be a good approximation of T in the sense of Definition 1.2. Then, for any smooth function φ on X we have

$$\lim_{n \rightarrow \infty} \int_Y (\alpha + dd^c v_n) \wedge f_*(\varphi f^*(T) \wedge f^*(T)) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

3) Assumptions are as in 2). Assume moreover that $f^*(T \wedge T)$ has no mass on $I(f)$. We write $f^*(T) = \Omega + dd^c u$, where Ω is a smooth closed $(1, 1)$ form and u is a quasi-psh function. Let $\Omega + dd^c u_j$ be a good approximation of $f^*(T) = \Omega + dd^c u$ in the sense of Definition 1.2. Then

$$\lim_{n \rightarrow \infty} \int_X \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).$$

Proof. 1) In this case, $f^*(T) = \Omega + dd^c u$, where Ω is a smooth closed $(1, 1)$ form and u is a quasi-psh function. Let u_j be a sequence of smooth quasi-psh functions decreasing to u . By the monotone convergence in Equation (2.6), we have

$$f_*(\varphi f^*(T) \wedge f^*(T)) = \lim_{n \rightarrow \infty} f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)).$$

By Theorem 1 in [11], $f_* f^* = Id$ for positive closed $(1, 1)$ and $(2, 2)$ currents. Since T satisfies Condition 1 and is in Class (\mathcal{A}) , for every smooth closed $(1, 1)$ form α we can apply Theorem 1.1 in [13] to obtain

$$f_*(\alpha \wedge f^*(T)) = f_*(f^*(f_* \alpha) \wedge f^*(T)) = f_*(f^*(f_*(\alpha) \wedge T)) = f_*(\alpha) \wedge T.$$

We now claim that

$$f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = f_*(\varphi) f_*(\Omega + dd^c u_j) \wedge f^*(T)$$

for every j . We choose $\Omega + dd^c u_j$ a good approximation for $f^*(T)$, in the sense of Definition 1.2. Therefore $\Omega + dd^c u_j + \epsilon \omega_X$ is positive for every j , here $\epsilon > 0$ is a constant. Since φ is bounded, $f_*(\varphi(\Omega + dd^c u_j + \epsilon \omega_X) \wedge f^*(T))$ is bounded by $f_*((\Omega + dd^c u_j + \epsilon \omega_X) \wedge f^*(T))$. The latter, as seen in the last paragraph, is the same as $f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge T$. It has no mass on $I(f^{-1})$. Therefore, $f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T))$ also has no mass on $I(f^{-1})$. Since $f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = f_*(\varphi) f_*(\Omega + dd^c u_j) \wedge T$ on $Y - I(f^{-1})$, we conclude that the equality holds on all of Y .

Recall that A is the finite set of points where T is not smooth. Since $\lim_{j \rightarrow \infty} f_*(\varphi)f_*(\Omega + dd^c u_j) = f_*(\varphi)f_*f^*(T) = f_*(\varphi)T$ on Y , we conclude that on $Y - A$

$$\lim_{j \rightarrow \infty} f_*(\varphi)f_*(\Omega + dd^c u_j) \wedge T = f_*(\varphi)T \wedge T.$$

By dimension reason, the above limit also holds on all of Y .

2) We need to show that

$$\lim_{n \rightarrow \infty} (\alpha + dd^c v_n) \wedge (f_*(\varphi)T \wedge T)^o = (f_*(\varphi)T \wedge T \wedge T)^o.$$

First, since $\alpha + dd^c v_n$ is smooth, we have

$$(\alpha + dd^c v_n) \wedge (f_*(\varphi)T \wedge T)^o = (f_*(\varphi)(\alpha + dd^c v_n) \wedge T \wedge T)^o.$$

Therefore, it suffices to show that any cluster point of $(f_*(\varphi)(\alpha + dd^c v_n) \wedge T \wedge T)^o$ has no mass on $I(f^{-1})$.

Since $\alpha + dd^c v_n$ is a good approximation of $\alpha + dd^c v$, there is a constant $\epsilon > 0$ such that $\alpha + dd^c v_n + \epsilon\omega_Y$ is positive for every n . We write

$$(f_*(\varphi)(\alpha + dd^c v_n) \wedge T \wedge T)^o = \mu_{1,n} - \mu_2,$$

Here

$$\begin{aligned} \mu_{1,n} &= (f_*(\varphi)(\alpha + dd^c v_n + \epsilon\omega_Y) \wedge T \wedge T)^o, \\ \mu_2 &= (f_*(\varphi)\epsilon\omega_Y \wedge T \wedge T)^o, \end{aligned}$$

are positive measures.

Since $\mu_{1,n}, \mu_2$ are bounded by the positive measures

$$\begin{aligned} \nu_{1,n} &= (\alpha + dd^c v_n + \epsilon\omega_Y) \wedge T \wedge T, \\ \nu_2 &= \epsilon\omega_Y \wedge T \wedge T, \end{aligned}$$

it suffices to show that ν_2 and any cluster point of $\nu_{1,n}$ have no mass on $I(f^{-1})$.

Since T is smooth outside a finite number of points and ω_Y is smooth, it is easy to see that ν_2 has no mass on $I(f^{-1})$.

The limit of $\nu_{1,n}$ is $(T + \epsilon)T \wedge T$ also has no mass on $I(f^{-1})$, since $T \wedge T \wedge T$ has no mass on $I(f^{-1})$ by Condition 3). Here, we use that monotone convergence holds, since $T \wedge T$ is smooth outside a finite number of points.

3) The proof is similar to the proof of 2). We write $T = \alpha + dd^c v$, where α is a smooth closed $(1, 1)$ form, and v is a quasi-psh function. Let $\alpha + dd^c v_n$ be a good approximation of T in the sense of Definition 1.2. Hence we can assume that $\alpha + dd^c v_n + \epsilon\omega_Y \geq 0$ for all n , here ϵ is a positive constant.

Then it is easy to see that

$$\lim_{j \rightarrow \infty} \int_X \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_X \varphi f^*(\alpha + dd^c v_n) \wedge (\Omega + dd^c u_j) \wedge f^*(T).$$

For each n, j then as in 1) and previous sections, we can show that

$$\int_X \varphi f^*(\alpha + dd^c v_n) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y (\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)).$$

Therefore, to prove 2), it suffices to show that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} (\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = (f_*(\varphi)T \wedge T \wedge T)^o.$$

Since $f : X - I(f) \rightarrow Y - I(f^{-1})$ is a pseudo-isomorphism, the above equality holds on $Y - I(f^{-1})$. Therefore, we only need to show that any cluster point of

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} (\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T))$$

has no mass on $I(f^{-1})$.

As in the proof of 1), we have that

$$(\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = (f_*(\varphi)(\alpha + dd^c v_n) \wedge f_*(\Omega + dd^c u_j) \wedge T)^o.$$

We write

$$(f_*(\varphi)(\alpha + dd^c v_n) \wedge f_*(\Omega + dd^c u_j) \wedge T)^o = \mu_{j,n} - \mu_{1,j,n} - \mu_{2,j,n},$$

where

$$\begin{aligned} \mu_{j,n} &= (f_*(\varphi)(\alpha + dd^c v_n + \epsilon \omega_Y) \wedge f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge T)^o, \\ \mu_{1,j,n} &= (f_*(\varphi) \epsilon \omega_Y \wedge f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge T)^o, \\ \mu_{2,j,n} &= (f_*(\varphi)(\alpha + dd^c v_n + \epsilon \omega_Y) \wedge \epsilon f_*(\omega_X) \wedge T)^o. \end{aligned}$$

Note that $\mu_{j,n}$, $\mu_{1,j,n}$, $\mu_{2,j,n}$ are positive measures and are bounded by the following positive measures

$$\begin{aligned} \nu_{j,n} &= (\alpha + dd^c v_n + \epsilon \omega_Y) \wedge f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge T, \\ \nu_{1,j,n} &= \epsilon \omega_Y \wedge f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge T, \\ \nu_{2,j,n} &= (\alpha + dd^c v_n + \epsilon \omega_Y) \wedge \epsilon f_*(\omega_X) \wedge T. \end{aligned}$$

Hence, it suffices to show that the following limits exist and have no mass on $I(f^{-1})$

$$\begin{aligned} &\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_{j,n}, \\ &\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_{1,j,n}, \\ &\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \nu_{2,j,n}. \end{aligned}$$

a) The first limit is

$$\begin{aligned} &\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} (\alpha + dd^c v_n + \epsilon \omega_Y) \wedge f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge T \\ &= \lim_{j \rightarrow \infty} f_*(\Omega + dd^c u_j + \epsilon \omega_X) \wedge (T + \epsilon \omega_Y) \wedge T \\ &= (T + f_*(\epsilon \omega_X)) \wedge (T + \epsilon \omega_Y) \wedge T. \end{aligned}$$

Here we used that T is smooth outside a finite number of points, hence monotone convergence holds. In the resulting limit:

- The term $T \wedge T \wedge T$ has no mass on $I(f^{-1})$ by Condition 3).
- The term $T \wedge \omega_Y \wedge T$ has no mass on $I(f^{-1})$ since T is smooth outside a point and ω_Y is smooth.
- The term $f_*(\omega_X) \wedge \omega_Y \wedge T$ has no mass on $I(f^{-1})$ since T is smooth outside a finite number of points, $f_*(\omega_X)$ has no mass on proper analytic subvarieties, and ω_Y is smooth.

- Now we show that the last term $f_*(\omega) \wedge T \wedge T$ has no mass on $I(f^{-1})$. By assumption, $f^*(T \wedge T)$ has no mass on $I(f)$, hence it is a positive current, and the positive measure $\omega_X \wedge f^*(T \wedge T)$ has no mass on $I(f)$. Therefore, the pushforward $f_*(\omega_X \wedge f^*(T \wedge T))$ is well-defined as a positive measure with no mass on $I(f^{-1})$. On $Y - I(f^{-1})$, then $f_*(\omega) \wedge T \wedge T = f_*(\omega_X \wedge f^*(T \wedge T))$. Therefore, $f_*(\omega) \wedge T \wedge T \geq f_*(\omega_X \wedge f^*(T \wedge T))$ on Y . Moreover, the masses of the two measures $f_*(\omega) \wedge T \wedge T$ and $f_*(\omega_X \wedge f^*(T \wedge T))$, which can be computed cohomologically, are the same. We conclude that $f_*(\omega) \wedge T \wedge T = f_*(\omega_X \wedge f^*(T \wedge T))$ on Y . Here we use the following properties of pseudo-isomorphisms in dimension 3: $f^*(\zeta) \cdot f^*(\eta) = f^*(\zeta \cdot \eta)$ (see [3]) for $\zeta \in H^{1,1}$ and $\eta \in H^{2,2}$.

Hence we conclude that the first limit has no mass on $I(f^{-1})$.

b) Using a similar argument, we have that the second and third limits also have no mass on $I(f^{-1})$, as wanted. \square

REFERENCES

- [1] **Note.** More relevant references will be added later.
- [2] G. Bassanelli, *A cut-off theorem for plurisubharmonic currents*, Forum Math. 6 (1994), no. 5, 567–595.
- [3] E. Bedford and K.-H. Kim, *Dynamics of pseudo-automorphisms of 3 spaces: periodicity versus positive entropy*, arXiv: 1101.1614.
- [4] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampere equation*, Invent. Math. 37 (1976), no. 1, 1–44.
- [5] J.-P. Demailly, *Complex analytic and differential geometry*, Online book, version of Thursday 10 September 2009.
- [6] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geometry 1. (1992), 361–409.
- [7] T-C Dinh and N. Sibony, *Regularization of currents and entropy*, Ann. Sci. Ecole Norm. Sup. (4), 37 (2004), no 6, 959–971.
- [8] T-C Dinh and N. Sibony, *Pull-back of currents by holomorphic maps*, Manuscripta Math. 123 (2007), no 3, 357–371.
- [9] J. E. Fornæss and N. Sibony, *Oka’s inequality for currents and applications*, Math. Ann. 301 (1995), 399–419.
- [10] Michel Meo, *Inverse image of a closed positive current by a surjective analytic map*, (in French), C. Acad. Sci. Paris Ser. I Math. 322 (1996), no 12, 1141–1144.
- [11] T.T. Truong, : *Some dynamical properties of pseudo-automorphisms in dimension 3*, accepted in Transactions of the American Mathematical Society. arXiv:1304.4100.
- [12] T.T. Truong, : *Pullback of currents by meromorphic maps*, preprint. To appear in Bulletin de Societe Mathematiques de France.
- [13] T.T. Truong, : *Pseudo-isomorphisms in dimension 3 and applications to complex Monge-Ampere operators*, arXiv: 1403.5325.

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE NY 13244
E-mail address: tutruong@syr.edu